Name (Last, First): ______________________________

Student ID: ______________________________

Circle your section:

201	Shin	8am	71 Evans	212	Lim	1pm	3105 Etcheverry
202	Cho	8am	75 Evans	213	Tanzer	2pm	35 Evans
203	Shin	9am	105 Latimer	214	Moody	2pm	81 Evans
204	Cho	9am	254 Sutardja Dai	215	Tanzer	3pm	206 Wheeler
205	Zhou	10am	254 Sutardja Dai	216	Moody	3pm	61 Evans
206	Theerakarn	10am	179 Stanley	217	Lim	8am	310 Hearst
207	Theerakarn	11am	179 Stanley	218	Moody	5pm	71 Evans
208	Zhou	11am	254 Sutardja Dai	219	Lee	5pm	3111 Etcheverry
209	Wong	12pm	3 Evans	220	Williams	12pm	289 Cory
210	Tabrizian	12pm	9 Evans	221	Williams	3pm	140 Barrows
211	Wong	1pm	254 Sutardja Dai	222	Williams	2pm	220 Wheeler

If none of the above, please explain: ______________________________

This is a closed book exam, no notes allowed. It consists of 6 problems, each worth 10 points. We will grade all 6 problems, and count your top 5 scores.

Problem	Maximum Score	Your Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total Possible	50	

Problem 1) Decide if the following statements are ALWAYS TRUE or SOMETIMES FALSE. You do not need to justify your answers. Write the full word **TRUE** or **FALSE** in the answer boxes of the chart. (Correct answers receive 2 points, incorrect answers or blank answers receive 0 points.)

Statement	1	2	3	4	5
Answer					

1) The matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are similar.

2) The matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ are similar.

3) For 2×2 matrices A and B, if $\mathbf{v}$ is an eigenvector of AB, then $B\mathbf{v}$ is an eigenvector of A.

4) If a 3×3 matrix A is diagonalizable with eigenvalues ± 1, then it is an orthogonal matrix.

5) If $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Problem 2) Indicate with an **X** in the chart all of the answers that satisfy the questions below. You do not need to justify your answers. It is possible that any number of the answers satisfy the questions. (A completely correct row of the chart receives 2 points, a partially correct row receives 1 point, but any incorrect X in a row leads to 0 points.)

	(a)	(b)	(c)	(d)	(e)
Question 1					
Question 2					
Question 3					
Question 4					
Question 5					

1) Which of the following conditions guarantees an $n \times n$ real matrix A is diagonalizable with real eigenvalues?

a) Every eigenvalue of A has an eigenvector.
b) There is a basis of $\mathbb{R}^n$ consisting of real eigenvectors for A.
c) $\det(A - \lambda I_n) = \lambda^n - \lambda^{n-1}$ and $\dim Nul(A) = 1$.
d) $\det(A - \lambda I_n) = \lambda^n - \lambda^{n-2}$ and $\dim Nul(A) = n - 2$.
e) The inverse of A is diagonalizable with real eigenvalues.

2) For what h is the matrix

$$\begin{bmatrix} 1 & -h^2 & 2h \\ 0 & 2h & h \\ 0 & 0 & h^2 \end{bmatrix}$$

diagonalizable with real eigenvalues?

a) $h = -2$ b) $h = -1$ c) $h = 0$ d) $h = 1$ e) $h = 2$

3) Which of the following linear transformations $T : P_2 \to P_2$ have rank 1?

$$a)\ T(p(x)) = p'(x) \quad b)\ T(p(x)) = p''(x) \quad c)\ T(p(x)) = (1+x)p'(x)$$

$$d)\ T(p(x)) = (1+x)p''(x) \quad e)\ T(p(x)) = (1+x)p(1)$$

4) Which of the following are a basis B of $\mathbb{R}^3$ so that for $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ we have $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$?

$$a) \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \quad b) \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} \quad c) \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ 9 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e) \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

5) Which of the following linear transformations $\mathbb{R}^2 \to \mathbb{R}^2, \mathbf{x} \mapsto A\mathbf{x}$ are given by an orthogonal matrix A?

- $a)$ Reflection across the line $x = y$.
- $b)$ Rotation by $\pi/4$ about the origin.
- $c)$ A shear transformation fixing the line $y = 0$.
- $d)$ Reflection across the line $x = y$ followed by reflection across the line $x = 0$.
- $e)$ Scaling by 2 followed by rotation by $\pi/4$ about the origin followed by scaling by $1/2$.

Problem 3) a) (4 points) Find the eigenvalues and a basis consisting of eigenvectors of

$$A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -3 & 1 \\ -3 & -3 & 1 \end{bmatrix}$$

b) (3 points) Find the coordinates of the vector

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

with respect to the basis of eigenvectors.

c) (3 points) Calculate $A^{2014}\mathbf{v}$.

Problem 4) a) (4 points) Calculate the matrix $[T]$ of the linear transformation

$$T : P_2 \to \mathbb{R}^3 \qquad T(p(x)) = \begin{bmatrix} p(1) \\ p'(0) - p'(1) \\ p'(0) + p'(1) \end{bmatrix}$$

with respect to the basis $B = \{1, 1+x, 1+x+x^2\}$ of P_2 and the standard basis of $\mathbb{R}^3$.

b) (4 points) Find bases of P_2 and $\mathbb{R}^3$ such that the matrix of T satisfies

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

c) (2 points) Find bases of P_2 and $\mathbb{R}^3$ such that the matrix of T^{-1} satisfies

$$[T^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 5) Consider the subspace W of $\mathbb{R}^4$ spanned by

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 4 \end{bmatrix}$$

a) (4 points) Find a nonzero vector $\mathbf{w}$ in W orthogonal to $\mathbf{u}$.

b) (3 points) Find the orthogonal projection of the vector

$$\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

to the subspace W.

c) (3 points) Find the orthogonal projection of the vector $\mathbf{y}$ to the orthogonal subspace $W^{\perp}$.

Problem 6) (10 points) Fill in the blanks (each worth 1/2 a point) in the proof of the following assertion.

Assertion. If a 2×2 matrix A satisfies $\det(A - \lambda I) = \lambda^2$, then $A^2 = 0$.

Proof. Since ________________________, the only ________________________ of A is 0.

There must be a corresponding ________________________, which we will call $\mathbf{v}$, because

________________________ implies ______________________ is not invertible, and therefore

________________________ must be nontrivial.

Choose any $\mathbf{w}$ linearly independent from $\mathbf{v}$. Thus the pair $\mathbf{v}, \mathbf{w}$ is a ______________________,

which we will call B, because $\mathbf{v}, \mathbf{w}$ must also ______________________. Thus there exist a, b

so that $A\mathbf{w} = a\mathbf{v} + b\mathbf{w}$. The matrix of A with respect to B is then

$$[A]_B = \begin{bmatrix} \_\_\_\_\_\_ & \_\_\_\_\_\_ \\ \_\_\_\_\_\_ & \_\_\_\_\_\_ \end{bmatrix}$$

We see that ____________________ $= 0$, since the ________________________ entries

of any ________________________ matrix are its ________________________.

Finally, let P_B be the matrix with columns $\mathbf{v}, \mathbf{w}$. Then $A =$ ________________________.

Since it is easy to see that ________________________ $= 0$, we also find

$$A^2 = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ = 0.$$